# REAL OPTION VALUE, CH 12
# MULTI-FACTOR SWITCHING OPTIONS

Switching options are often embedded in facilities, or situations, sometimes developed with the imagination and initiative of the participant, or manager. For instance, Miss Lucy Steele obtained the best of (her perceived) two outcomes in marrying Robert Ferrars, after he had received the family fortune, abandoning her previous engagement to Edward Ferrars[1].

This chapter presents three basic two factor multiple switching option models: (i) switching to the highest price output; (ii) switching from an operating state with an option to suspend operations, or from a suspended state to an operating state, when both output price and input cost are stochastic; and (iii) switching to the lowest cost input. As a simplification (and reduction in the number of equations required for a solution), a single switch, such as from operating to suspended, is also considered. The model for the best of two outputs is adapted from Dockendorf and Paxson (2011), for suspension and restart options from Adkins and Paxson (2012), and for two inputs from Adkins and Paxson (2011).

## 1 GENERAL SWITCHING OPTIONS

When is the right time for an operator of a flexible facility to switch back and forth between two possible outputs or inputs in order to maximise value when switching costs are taken into account? Which factors should be monitored in making these decisions? How much should an investor pay for such a flexible operating asset? What are the strategy implications for the operator, investor and possibly for policy makers?

Flexible production and processing facilities are typically more expensive to operate, and with a higher initial investment cost, than inflexible facilities. One problem is that one part of the flexible facility, which requires an additional

[1] Jane Austen, Sense and Sensibility, 1811. Edward Ferrars's inheritance is settled on his younger brother Robert, when his mother hears of Lucy's engagement to Edward. Lucy promptly transfers her affection, following the money, apparently a single switch.

investment cost, might be idle at times. Investing in a facility which is not productive all the time seems counter-intuitive at first glance. What is frequently misunderstood is that the additional option value through "operating flexibility" (Trigeorgis and Mason, 1987) may have significant value in uncertain markets and when there is less than perfect correlation between input factors, or between possible outputs, or indeed between inputs and outputs. Examples of flexible assets include shipping (combination carriers), the chemical industry (flexible fertilizer plants), electricity generation (system switching from coal to natural gas), and real estate (multiple property uses).

The traditional approach to determine switching boundaries between two operating modes is to discount future cash flows and use Jevons-Marshallian present value triggers. This methodology does not fully capture the option value which may arise due to the uncertainty in future input or output prices. The value of waiting to gain more information on future price or cost developments, and consequently on the optimal switching triggers can be best viewed in a real options framework.

Conceptually, the switch between two volatile assets or commodities can be modelled as an exchange option. Margrabe (1978) and McDonald and Siegel (1986) model European and American perpetual exchange options, respectively, which are linear homogeneous in the underlying stochastic variables. An analytical model for flexible production capacity is presented by He and Pindyck (1992), where switching costs and product-specific operating costs are ignored, thereby eliminating the components which would lead to a non-linearity of the value function in the underlying processes. Brekke and Schieldrop (2000) also assume costless switching in their study on the value of operating flexibility between two stochastic input factors, in which they determine the optimal investment timing for a flexible technology in comparison to a technology that does not allow input switching. Adkins and Paxson (2011) present quasi-analytical solutions to input switching options, where two-factor functions are not homogeneous of degree one, and thus dimension reducing techniques used in McDonald and Siegel (1986) and Paxson and Pinto (2005) are not available. Other approaches, such as Song et al. (2010), consider the net profits (revenue less cost) or spreads in the respective operating state to be stochastic, reducing two factors to one.

Brennan and Schwartz (1985) consider switching states from idle to operating, operating to suspension, and then back, based on one stochastic factor. Paxson (2005) extends the solution for up to eight different state options, each with a distinct trigger, but for only one stochastic factor.

Geltner, Riddiough and Stojanovic (1996) develop a framework for a perpetual option on the best of two underlying assets, applied to the case of two alternative uses for properties, and provide a comprehensive discussion of relevant assumptions for such a contingent-claims problem. Childs, Riddiough and Triantis (1996) extend this model to allow for redevelopment or switching between alternative uses.

The next section presents two real option models for an asset with switching opportunities between two outputs with uncertain prices, taking into account switching costs and operating costs. The first model is a quasi-analytical solution for multiple switching among the best of two outputs; the second for single one-way switching.

## 2 Multiple Output Switching

### *2.1 Assumptions*

Consider a flexible facility which can be used to produce one of two different outputs by switching between operating modes. Assume the prices of the two outputs, x and y, are stochastic and correlated and follow geometric Brownian motion (gBm):

$$dx = (\mu_x - \delta_x)x\,dt + \sigma_x\,x\,dz_x \tag{1}$$

$$dy = (\mu_y - \delta_y)y\,dt + \sigma_y\,y\,dz_y \tag{2}$$

with the notations:

$\mu$ Required return on the output

$\delta$ Convenience yield of the output

$\sigma$ Volatility of the output

dz Wiener process (stochastic element)

$\rho$ Correlation between the two output prices: $dz_x\,dz_y$ / dt

The instantaneous cash flow in each operating mode is the respective commodity price of the output product less unit operating cost, assuming production of one (equivalent) unit per annum, $(x - c_x)$ in operating mode '1' and $(y - c_y)$ in operating mode '2'. The operating costs $c_x$ and $c_y$ are per unit produced. A switching cost of $S_{12}$ is incurred when switching from operating mode '1' to '2', and $S_{21}$ for switching back. The appropriate discount rate is r for non- stochastic elements, such as constant operating costs. For convenience and simplicity, assume that the appropriate discount rate for stochastic variables is $\delta$ which is equal to $\mu$-r.

Further assumptions are that the operating costs are deterministic and constant, the lifetime of the asset is infinite, and the company is not restricted in the product mix choice because of selling commitments. Moreover, the typical assumptions of real options theory apply, with interest rates, convenience yields, volatilities and correlation constant over time.

### *2.2 Quasi-analytical Solution for Continuous Switching*

The asset value with opportunities to continuously switch between the two operating modes is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let $V_1$ be the asset value in operating mode '1', producing output x, and $V_2$ the asset value in operating mode '2', producing output y accordingly. The switching options depend on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by the following partial differential equations (PDE):

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_1}{\partial x} + (r-\delta_Y)y\frac{\partial V_1}{\partial y} - rV_1 + (x - c_X) = 0 \tag{3}$$

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_2}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_2}{\partial x} + (r-\delta_Y)y\frac{\partial V_2}{\partial y} - rV_2 + (y - c_y) = 0 \tag{4}$$

Two-factor problems which are linear homogeneous, i.e. $\mathrm{V}(\lambda \cdot \mathrm{x};\lambda \cdot \mathrm{y}) = \lambda \cdot \mathrm{V}(\mathrm{x};\mathrm{y})$, can typically be solved analytically by substitution of variables, so that the PDE can be reduced to a one-factor differential equation. An example of this is the perpetual American exchange option in McDonald and Siegel (1986). With constant switching cost, operating cost and multiple switching, the problem is no longer homogenous of degree one and the dimension reducing technique cannot be used.

Dockendorf and Paxson (2011) derive a quasi-analytical solution for a similar type of two-factor non-homogeneous problem. For two outputs, the PDE are satisfied by the following general solutions:

$$V_1(x,y)=\frac{x}{\delta_x}-\frac{c_x}{r}+Ax^{\beta_{11}}y^{\beta_{12}} \tag{5}$$

where $\beta_{11}$ and $\beta_{12}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1)+\tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1)+\rho\sigma_x\sigma_y\beta_{11}\beta_{12}+\beta_{11}(r-\delta_x)+\beta_{12}(r-\delta_y)-r=0, \tag{6}$$

and

$$V_2(x,y)=\frac{y}{\delta_y}-\frac{c_y}{r}+Bx^{\beta_{21}}y^{\beta_{22}} \tag{7}$$

where $\beta_{21}$ and $\beta_{22}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{21}(\beta_{21}-1)+\tfrac{1}{2}\sigma_y^2\beta_{22}(\beta_{22}-1)+\rho\sigma_x\sigma_y\beta_{21}\beta_{22}+\beta_{21}(r-\delta_x)+\beta_{22}(r-\delta_y)-r=0 \tag{8}$$

Since the option to switch from x to y decreases with x and increases with y, $\beta_{11}$ must be negative and $\beta_{12}$ positive. Likewise, $\beta_{21}$ must be positive and $\beta_{22}$ negative. Switching between the operating modes always depends on the level of both x and y. At the switching points ($x_{12}$, $y_{12}$) and ($x_{21}$, $y_{21}$), the asset value in the current operating mode must be equal to the asset value in the alternative operating mode net of switching cost. These value matching conditions are:

$$A{x_{12}}^{\beta_{11}}{y_{12}}^{\beta_{12}}+\frac{x_{12}}{\delta_x}-\frac{c_x}{r}=B{x_{12}}^{\beta_{21}}{y_{12}}^{\beta_{22}}+\frac{y_{12}}{\delta_y}-\frac{c_y}{r}-S_{12} \tag{9}$$

$$A{x_{21}}^{\beta_{11}}{y_{21}}^{\beta_{12}}+\frac{x_{21}}{\delta_x}-\frac{c_x}{r}-S_{21}=B{x_{21}}^{\beta_{21}}{y_{21}}^{\beta_{22}}+\frac{y_{21}}{\delta_y}-\frac{c_y}{r} \tag{10}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11}A{x_{12}}^{\beta_{11}-1}{y_{12}}^{\beta_{12}}+\frac{1}{\delta_x}=\beta_{21}B{x_{12}}^{\beta_{21}-1}{y_{12}}^{\beta_{22}} \tag{11}$$

$$\beta_{12}A{x_{12}}^{\beta_{11}}{y_{12}}^{\beta_{12}-1}=\beta_{22}B{x_{12}}^{\beta_{21}}{y_{12}}^{\beta_{22}-1}+\frac{1}{\delta_y} \tag{12}$$

$$\beta_{11}A{x_{21}}^{\beta_{11}-1}{y_{21}}^{\beta_{12}}+\frac{1}{\delta_x}=\beta_{21}B{x_{21}}^{\beta_{21}-1}{y_{21}}^{\beta_{22}} \tag{13}$$

$$\beta_{12}A{x_{21}}^{\beta_{11}}{y_{21}}^{\beta_{12}-1}=\beta_{22}B{x_{21}}^{\beta_{21}}{y_{21}}^{\beta_{22}-1}+\frac{1}{\delta_y} \tag{14}$$

There are only 8 equations, (6) and (8) - (14), for 10 unknowns, $\beta_{11}$, $\beta_{12}$, $\beta_{21}$, $\beta_{22}$, A, B, $x_{12}$, $y_{12}$, $x_{21}$, $y_{21}$, so there is no completely analytical solution. Yet, for every value of x, there has to be a corresponding value of y when switching should occur, ($x_{12}$, $y_{12}$) from x to y and ($x_{21}$, $y_{21}$) from y to x. So a quasi-analytical solution can be found by assuming values for x, which then solves the set of simultaneous equations for all remaining variables, given that $x = x_{12} = x_{21}$. This procedure is repeated for many values of x, providing the corresponding option values and the switching boundaries.

The spread between the two switching boundaries can be viewed in term of the wedges, defined below.

$$\frac{y_{12}}{\delta_y} \cdot \Omega_{y_{12}} - \frac{x_{12}}{\delta_x} \cdot \Omega_{x_{12}} = S_{12} \tag{15}$$

$$\frac{x_{21}}{\delta_x} \cdot \Omega_{x_{21}} - \frac{y_{21}}{\delta_y} \cdot \Omega_{y_{21}} = S_{21} \tag{16}$$

The Marshallian rule is satisfied when all wedges ($\Omega_{x12}$, $\Omega_{x21}$, $\Omega_{y12}$, $\Omega_{y21}$) are equal to one. The wedges for the real option model are:

$$\Omega_{x_{12}} = \Omega_{x_{21}} = 1 - \frac{\beta_{12} - \beta_{22}}{\beta_{12}\,\beta_{21} - \beta_{11}\,\beta_{22}} \tag{17}$$

$$\Omega_{y_{12}} = \Omega_{y_{21}} = 1 - \frac{\beta_{21} - \beta_{11}}{\beta_{12}\,\beta_{21} - \beta_{11}\,\beta_{22}} \tag{18}$$

Since $\beta_{12}$ and $\beta_{21}$ are positive and $\beta_{11}$ and $\beta_{22}$ are negative and the denominator of (17) and (18) needs to be positive to justify the option values, the wedges are less than one. This demonstrates that the switching hysteresis (band of inaction, no switching) is larger than suggested by the Marshallian rule.

It can be seen by rearranging (9) and (10) that the net difference in the switching options including the value of outputs at the triggers is the sum of the switching cost and the difference in the present value of operating costs, $S_{12} + \left(\frac{c_y - c_x}{r}\right)$ and $S_{21} - \left(\frac{c_y - c_x}{r}\right)$, respectively.

## 2.3 *Quasi-analytical Solution for One-Way Switching*

The solution for the asset value with a one-way switching option from the above model with continuous switching is straight-forward. Assuming $c_y \geq c_x$, the American perpetual option to switch from x to y can be determined. The asset value $V_{1S}$ is given by (5) with the characteristic root equation (22), and $V_{2S}$ is given by (7) with B=0, thereby eliminating the option to switch back. Applying the same solution procedure as before, a quasi-analytical solution is obtained.

$$A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}} + \frac{x_{12}}{\delta_x} - \frac{c_x}{r} = \frac{y_{12}}{\delta_y} - \frac{c_y}{r} - S_{12} \tag{19}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}^{\beta_{11}-1} y_{12}^{\beta_{12}} + \frac{1}{\delta_x} = 0 \tag{20}$$

$$\beta_{12} A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}-1} - \frac{1}{\delta_y} = 0 \tag{21}$$

where $\beta_{11}$ and $\beta_{12}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1) + \tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1) + \rho\sigma_x\sigma_y\beta_{11}\beta_{12} + \beta_{11}(r-\delta_x) + \beta_{12}(r-\delta_y) - r = 0 \tag{22}$$

The characteristic root equation (22) together with value matching condition (19) and smooth pasting conditions (20) and (21) represents the system of 4 equations, while there are 5 unknowns, $\beta_{11}$, $\beta_{12}$, A, $x_{12}$, $y_{12}$.

**Numerical Illustrations**

Here are illustrative results for the multiple and single output switch models, assuming current operating costs are half of current gross revenue for each output. Figure 1 shows that the option factors A and B are positive, $\beta_{11}$ and $\beta_{22}$ are negative and $\beta_{12}$ and $\beta_{21}$ are positive, thereby fulfilling the requirements from the theoretical model. The system of value matching conditions, smooth pasting conditions and characteristic root equations is fully satisfied.

Figure 1

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Multiple American Perpetual Output Switch Option** | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | OUTPUT x | x | 100 | | | | | | |
| 4 | OUTPUT y | y | 100 | | | | | | |
| 5 | Convenience yield of x | $\delta_x$ | 0.04 | | | | | | |
| 6 | Convenience yield of y | $\delta_y$ | 0.04 | | | | | | |
| 7 | Volatility of x | $\sigma_x$ | 0.40 | | | | | | |
| 8 | Volatility of y | $\sigma_y$ | 0.30 | | | | | | |
| 9 | Correlation x with y | $\rho$ | 0.50 | | | | | | |
| 10 | Risk-free interest rate | r | 0.05 | | | | | | |
| 11 | Operating cost for x | $c_x$ | 50 | | | | | | |
| 12 | Operating cost for y | $c_y$ | 50 | | | | | | |
| 13 | | | | | | | | | |
| 14 | | | | | | | | | |
| 15 | Switching cost from x to y | $S_{12}$ | 50 | | | | | | |
| 16 | Switching cost from y to x | $S_{21}$ | 70 | | | | | | |
| 17 | PV of revenues x | X | 1,500 | | | | | | |
| 18 | PV of revenues y | Y | 1,500 | | | | | | |
| 19 | Switching boundary x to y | $x_{12}$ | 100 | | | | | | |
| 20 | Switching boundary y to x | $x_{21}$ | 100 | | | | | | |
| 21 | | SOLUTION | | OPTION VALUE | | | | | |
| 22 | Asset value in operating mode '1' | **$V_1$(x,y)** | **2,780.37** | **1,280** | | | | | |
| 23 | Asset value in operating mode '2' | **$V_2$(x,y)** | **2,781.76** | **1,282** | | | | | |
| 24 | | A | 10.17 | | | | | | |
| 25 | | B | 9.76 | | | | | | |
| 26 | Switching boundary x to y | $y_{12}$ (x) | **150.21** | | | | | | |
| 27 | Switching boundary y to x | $y_{21}$ (x) | **53.18** | | | | | | |
| 28 | Solution quadrant | $\beta_{11}$ | -0.4012 | *must be negative* | | | | | |
| 29 | Solution quadrant | $\beta_{12}$ | 1.4512 | *must be positive* | | | | | |
| 30 | Solution quadrant | $\beta_{21}$ | 1.4229 | *must be positive* | | | | | |
| 31 | Solution quadrant | $\beta_{22}$ | -0.3636 | *must be negative* | | | | | |
| 32 | | EQUATIONS | | | | | | | |
| 33 | Value matching 1 | EQ9 | 0.000 | | | | | | |
| 34 | Value matching 2 | EQ10 | 0.000 | | | | | | |
| 35 | Smooth pasting 1A | EQ11 | 0.000 | | | | | | |
| 36 | Smooth pasting 1B | EQ12 | 0.000 | | | | | | |
| 37 | Smooth pasting 2A | EQ13 | 0.000 | | | | | | |
| 38 | Smooth pasting 2B | EQ14 | 0.000 | | | | | | |
| 39 | Solution quadrant 1 | EQ6 | 0.000 | | | | | | |
| 40 | Solution quadrant 2 | EQ8 | 0.000 | | | | | | |
| 41 | | Sum | 0.000 | | | | | | |
| 42 | | SOLVER: SET C40=0, CHANGING C24:C31 | | | | | | | |
| 43 | | Δ | 1.9190 | C29*C30-C28*C31 | | | | | |
| 44 | | Wedge x | 0.0543 | 1-(C29-C31)/C43 | | | | | |
| 45 | | Wedge y | 0.0495 | 1-(C30-C28)/C43 | | | | | |
| 46 | | EQ 15 | 0.0000 | (C26/C6)*C45-(C19/C5)*C44-C15 | | | | | |
| 47 | | EQ 16 | 0.0000 | (C20/C5)*C44-(C27/C6)*C45-C16 | | | | | |
| 48 | EQ9 | (C24*C19^C28*C26^C29+C19/C5-C11/C10-C25*C19^C30*C26^C31-C26/C6+C12/C10+C15) | | | | | | | |
| 49 | EQ10 | (C24*C20^C28*C27^C29+C20/C5-C11/C10-C25*C20^C30*C27^C31-C27/C6+C12/C10-C16) | | | | | | | |
| 50 | EQ11 | (C28*C24*C19^(C28-1)*C26^C29+1/C5-C30*C25*C19^(C30-1)*C26^C31) | | | | | | | |
| 51 | EQ12 | (C29*C24*C19^C28*C26^(C29-1)-C31*C25*C19^C30*C26^(C31-1)-1/C6) | | | | | | | |
| 52 | EQ13 | (C28*C24*C20^(C28-1)*C27^C29+1/C5-C30*C25*C20^(C30-1)*C27^C31) | | | | | | | |
| 53 | EQ14 | (C29*C24*C20^C28*C27^(C29-1)-C31*C25*C20^C30*C27^(C31-1)-1/C6) | | | | | | | |
| 54 | EQ6 | 0.5*C7^2*C28*(C28-1)+0.5*C8^2*C29*(C29-1)+C9*C7*C8*C28*C29+C28*(C10-C5)+C29*(C10-C6)-C10 | | | | | | | |
| 55 | EQ8 | 0.5*C7^2*C30*(C30-1)+0.5*C8^2*C31*(C31-1)+C9*C7*C8*C30*C31+C30*(C10-C5)+C31*(C10-C6)-C10 | | | | | | | |
| 56 | SPREAD | | 97.04 | | | | | | |
| 57 | $V_1$(x,y) EQ 5 | C3/C5-C11/C10+C24*C3^C28*C4^C29 | | | | | | | |
| 58 | $V_2$(x,y) EQ 7 | C4/C6-C12/C10+C25*C3^C30*C4^C31 | | | | | | | |

The asset values are given in both operating modes, $V_1$ and $V_2$, and the level of y is indicated when it is optimal to switch from x to y ($y_{12}$) and vice versa ($y_{21}$). With x and y having the same initial values and the same convenience yields, the asset value with no switching is identical in both operating modes when the operating cost is the same. Higher operating costs reduce the asset value. When operating costs are 50, the asset value $V_1$ with continuous switching opportunities is valued at 2780 if the incumbent is $x_{12}$=100 with a volatility of 40% according to the quasi-analytical solution. The switching option value is the difference between the asset value and the value with no switching option, 2780-1500=1280, and 2782-1500=1282 for $V_2$. The option to switch between the two operating modes adds about 85% to the inflexible asset value. Switching to output y is justified if y increases to 50% higher than output x, and back to x, if the y output price falls to half of the x price. The spread between $y_{12}$ and $y_{21}$ is due to switching costs and stochastic elements, and increases with high volatilities and low correlation, following real options theory. It should be noted that changing x also changes the switching boundaries $y_{12}$ and $y_{21}$, and that the switching boundaries $x_{12}$ and $x_{21}$ for a given level of y can be determined in a similar way. The fact that $y_{12}$ and $y_{21}$ are not symmetrical to x = 100 is primarily due to the log-normality of the prices, and further due to $S_{12} \neq S_{21}$ and $\sigma_x \neq \sigma_y$.

Figure 2

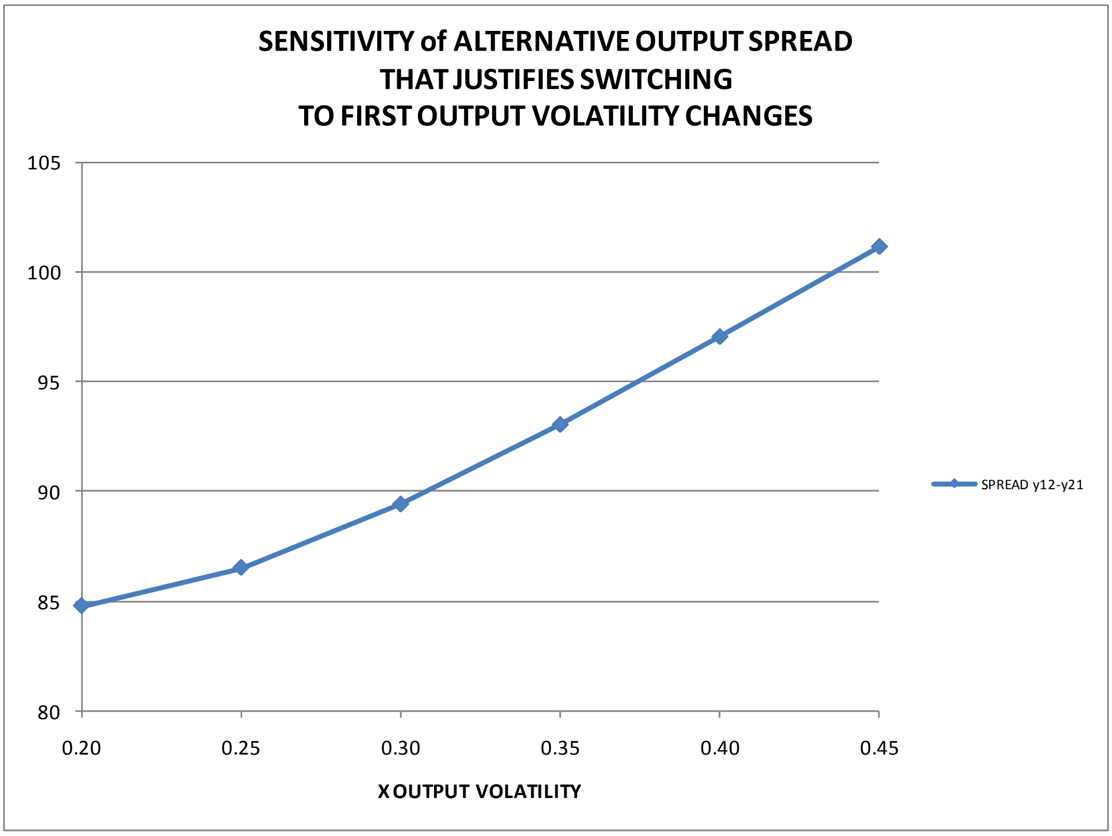


Figure 2 illustrates the sensitivity of the switching boundaries of the quasi-analytical solution for continuous switching to changes in x output price volatility. Switching boundaries are further apart when volatilities are higher. This is consistent with general real option theory because uncertainty is taken into account which delays switching in order to gain more information. In contrast to this, the Marshallian rule stipulates that switching is optimal as soon as the present value of expected cash flows after switching exceeds the present value of expected cash flows before switching by the switching cost.

Figure 3

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Continuous American Perpetual SINGLE SWITCH Option** | | | | | | | |
| 2 | ONE WAY SWITCH FROM OUTPUT X TO Y | | | | | | | |
| 3 | OUTPUT X | x | 100 | | | | | |
| 4 | OUTPUT Y | y | 100 | | | | | |
| 5 | Convenience yield of x | $\delta_x$ | 0.04 | | | | | |
| 6 | Convenience yield of y | $\delta_y$ | 0.04 | | | | | |
| 7 | Volatility of x | $\sigma_x$ | 0.40 | | | | | |
| 8 | Volatility of y | $\sigma_y$ | 0.30 | | | | | |
| 9 | Correlation x with y | $\rho$ | 0.50 | | | | | |
| 10 | Risk-free interest rate | r | 0.05 | | | | | |
| 11 | Operating cost for x | $c_x$ | 50 | | | | | |
| 12 | Operating cost for y | $c_y$ | 50 | | | | | |
| 13 | Switching cost from x to y | $S_{12}$ | 50 | | | | | |
| 14 | | | | | | | | |
| 15 | PV of revenues x | X | 1,500 | LHS EQ5 | | | | |
| 16 | PV of revenues y | Y | 1,500 | LHS EQ 7 | | | | |
| 17 | Switching boundary x to y | $x_{12}$ | 100 | | | | | |
| 18 | | | | | | | | |
| 19 | | SOLUTION | | OPTION VALUE | | | | |
| 20 | Asset value in operating mode '1' | **$V_1$(x,y)** | **2,528.98** | **1,029** | | | | |
| 21 | Asset value in operating mode '2' | **$V_2$(x,y)** | **1,500.00** | 0 | | | | |
| 22 | | A | 9.89 | | | | | |
| 23 | Switching boundary x to y | $y_{12}$ (x) | **337.04** | | | | | |
| 24 | Solution quadrant | $\beta_{11}$ | -0.4255 | *must be negative* | | | | |
| 25 | Solution quadrant | $\beta_{12}$ | 1.4340 | *must be positive* | | | | |
| 26 | | EQUATIONS | | | | | | |
| 27 | Value matching 1 | EQ 19 | 0.000 | | | | | |
| 28 | Smooth pasting 1A | EQ 20 | 0.000 | | | | | |
| 29 | Smooth pasting 1B | EQ 21 | 0.000 | | | | | |
| 30 | Solution quadrant 1 | EQ 22 | 0.000 | | | | | |
| 31 | | Sum | 0.000 | | | | | |
| 32 | | SOLVER: SET C31=0, CHANGING C22:C25 | | | | | | |
| 33 | EQ 19 | (C22*C17^C24*C23^C25+C17/C5-C11/C10-C23/C6+C12/C10+C13) | | | | | | |
| 34 | EQ 20 | (C24*C22*C17^(C24-1)*C23^C25+1/C5) | | | | | | |
| 35 | EQ 21 | (C25*C22*C17^C24*C23^(C25-1)-1/C6) | | | | | | |
| 36 | EQ 22 | 0.5*C7^2*C24*(C24-1)+0.5*C8^2*C25*(C25-1)+C9*C7*C8*C24*C25+C24*(C10-C5)+C25*(C10-C6)-C10 | | | | | | |
| 37 | SPREAD | | **237.04** | | | | | |
| 38 | $V_1$(x,y) EQ 5 | C3/C5-C11/C10+C22*C3^C24*C4^C25 | | | | | | |
| 39 | $V_2$(x,y) EQ 7 | C4/C6-C12/C10 | | | | | | |

When switching is only possible from x to y but not vice versa, the switching trigger $y_{12S}$ is much (224%) higher as shown in Figure 3 because the decision cannot be reversed. The asset value $V_{1S}$ is 9% lower compared to multiple switching.

### *3 Quasi-analytical Solution for Continuous Input-Output Switching*

The asset value with opportunities to continuously switch between an operating mode and a suspended mode (when both inputs and outputs are stochastic) is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let $V_1$ be the asset value in operating mode '1', producing output x at input cost y, and $V_2$ the asset value in a suspension mode '2'. Following Adkins and Paxson (2012) the switching options depend on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by the following partial differential equations:

(23)

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_1}{\partial x} + (r-\delta_Y)y\frac{\partial V_1}{\partial y} - rV_1 + (x-y) = 0$$

(24)

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_2}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_2}{\partial x} + (r-\delta_Y)y\frac{\partial V_2}{\partial y} - rV_2 = 0$$

The operating mode has the production income (x-y) and the option to suspend; the suspension mode has only the option to re-start operations. For stochastic outputs and inputs, the partial differential equations are satisfied by the following general solutions:

$$V_1(x,y) = \frac{x}{\delta_x} - \frac{y}{\delta_y} + Ax^{\beta_{11}} y^{\beta_{12}} \quad (25)$$

where $\beta_{11}$ and $\beta_{12}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1) + \tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1) + \rho\sigma_x\sigma_y\beta_{11}\beta_{12} + \beta_{11}(r-\delta_x) + \beta_{12}(r-\delta_y) - r = 0, \quad (26)$$

and

$$V_2(x,y) = Bx^{\beta_{21}} y^{\beta_{22}} \quad (27)$$

where $\beta_{21}$ and $\beta_{22}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{21}(\beta_{21}-1) + \tfrac{1}{2}\sigma_y^2\beta_{22}(\beta_{22}-1) + \rho\sigma_x\sigma_y\beta_{21}\beta_{22} + \beta_{21}(r-\delta_x) + \beta_{22}(r-\delta_y) - r = 0$$

(28)

Since the option to switch from operating to suspension decreases with x and increases with y, $\beta_{11}$ must be negative and $\beta_{12}$ positive. Likewise, $\beta_{21}$ must be positive and $\beta_{22}$ negative. Switching between the operating and suspension modes always depends on the level of both x and y. At the switching points ($x_{12}$, $y_{12}$) and ($x_{21}$, $y_{21}$), the asset value in the current operating mode must be equal to the asset value in the alternative operating mode net of switching cost. These value matching conditions are:

$$V_1\left(x_{12}, y_{12}\right) = V_2(x_{12}, y_{12}) - S_{12}$$

$$V_2\left(x_{21}, y_{21}\right) = V_1(x_{21}, y_{21}) - S_{21}$$

$$A\, x_{12}^{\ \beta_{11}}\, y_{12}^{\ \beta_{12}} + \frac{x_{12}}{\delta_x} - \frac{y_{12}}{\delta_y} = B\, x_{12}^{\ \beta_{21}}\, y_{12}^{\ \beta_{22}} - S_{12} \tag{29}$$

$$B\, x_{21}^{\ \beta_{21}}\, y_{21}^{\ \beta_{22}} = A x_{21}^{\ \beta_{11}}\, y_{21}^{\ \beta_{12}} + \frac{x_{21}}{\delta_x} - \frac{y_{21}}{\delta_y} - S_{21} \tag{30}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}^{\ \beta_{11}-1}\, y_{12}^{\ \beta_{12}} + \frac{1}{\delta_x} = \beta_{21} B\, x_{12}^{\ \beta_{21}-1}\, y_{12}^{\ \beta_{22}} \tag{31}$$

$$\beta_{12}\, A\, x_{12}^{\ \beta_{11}}\, y_{12}^{\ \beta_{12}-1} - \frac{1}{\delta_y} = \beta_{22}\, B\, x_{12}^{\ \beta_{21}}\, y_{12}^{\ \beta_{22}-1} \tag{32}$$

$$\beta_{21}\, B\, x_{21}^{\ \beta_{21}-1}\, y_{21} = \beta_{11}\, A\, x_{21}^{\ \beta_{11}-1}\, y_{21}^{\ \beta_{12}} + \frac{1}{\delta_x} \tag{33}$$

$$\beta_{22} B x_{21}^{\ \beta_{21}}\, y_{21}^{\ \beta_{22}-1} = \beta_{12} A x_{21}^{\ \beta_{11}}\, y_{21}^{\ \beta_{12}-1} - \frac{1}{\delta_y} \tag{34}$$

There are only 8 equations, (26) and (28- 34), for 10 unknowns, $\beta_{11}$, $\beta_{12}$, $\beta_{21}$, $\beta_{22}$, A, B, $x_{12}$, $y_{12}$, $x_{21}$, $y_{21}$, so there is no completely analytical solution. Yet, for every value of x, there has to be a corresponding value of y when switching should occur, ($x_{12}$, $y_{12}$) and ($x_{21}$, $y_{21}$). So a quasi-analytical solution can be found by assuming values for x, which then solves the set of simultaneous equations for all remaining variables, given that $x = x_{12} = x_{21}$. This procedure is repeated for many values of x, providing the corresponding option values and the switching boundaries.

**Single Switch**

The solution for the asset value with a one-way switching option from the above model with multiple switching is straight-forward. This one-way switch constitutes an abandonment option, where the switching cost is the abandonment cost. The asset value $V_{1S}$ is given by (25) with the characteristic root equation (38), and $V_{2S}$ is given by (27) with B=0, thereby eliminating the option to switch back. Applying the same solution procedure as before, a quasi-analytical solution is obtained.

$$A\, x_{12}^{\ \beta_{11}}\, y_{12}^{\ \beta_{12}} + \frac{x_{12}}{\delta_x} - \frac{y_{12}}{\delta_y} + S_{12} = 0 \tag{35}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}^{\beta_{11}-1} y_{12}^{\beta_{12}} + \frac{1}{\delta_x} = 0 \tag{36}$$

$$\beta_{12} A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}-1} - \frac{1}{\delta_y} = 0 \tag{37}$$

where $\beta_{11}$ and $\beta_{12}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1)+\tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1)+\rho\sigma_x\sigma_y\beta_{11}\beta_{12}+\beta_{11}(r-\delta_x)+\beta_{12}(r-\delta_y)-r=0 \tag{38}$$

The characteristic root equation (38) together with value matching condition (35) and smooth pasting conditions (36) and (37) represents the system of 4 equations, while there are 5 unknowns, $\beta_{11}$, $\beta_{12}$, A, $x_{12}$, $y_{12}$.

Figure 4

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Continuous American Perpetual IN-OUT Exchange Option Non-homogeneous** | | | | | | | | | |
| 2 | | INPUT | | | IN_OUT_JAN2011 | | | | | |
| 3 | PRICE | x | 100 | | | | | | | |
| 4 | COST | y | 50 | | | | | | | |
| 5 | Convenience yield of x | $\delta_x$ | 0.04 | | | | | | | |
| 6 | Convenience yield of y | $\delta_y$ | 0.04 | | | | | | | |
| 7 | Volatility of x | $\sigma_x$ | 0.40 | | | | | | | |
| 8 | Volatility of y | $\sigma_y$ | 0.30 | | | | | | | |
| 9 | Correlation x with y | $\rho$ | 0.50 | | | | | | | |
| 10 | Risk-free interest rate | r | 0.05 | | | | | | | |
| 11 | | | | | | | | | | |
| 12 | | | | | | | | | | |
| 13 | | | | | | | | | | |
| 14 | | | | | | | | | | |
| 15 | Switching cost from x to y | $S_{12}$ | 50 | | | | | | | |
| 16 | Switching cost from y to x | $S_{21}$ | 70 | | | | | | | |
| 17 | | | | | | | | | | |
| 18 | | | | | | | | | | |
| 19 | Switching boundary OP to SHUT | $x_{12}$ | 100 | | | | | | | |
| 20 | Switching boundary SHUT to OP | $x_{21}$ | 100 | | | | | | | |
| 21 | | SOLUTION | | OPTION | OPERATING | | | | | |
| 22 | Asset value in operating mode | **$V_1$(x,y)** | **1718.27** | **468.27** | **1250.00** | | | | | |
| 23 | Asset value in shut down mode | **$V_2$(x,y)** | **1649.16** | **1649.16** | | | | | | |
| 24 | | A | 10.17 | | | | | | | |
| 25 | | B | 9.76 | | | | | | | |
| 26 | Switching boundary OP to SHUT | $y_{12}$ (x) | **150.21** | | | | | | | |
| 27 | Switching boundary SHUT to OP | $y_{21}$ (x) | **53.18** | | | | | | | |
| 28 | Solution quadrant | $\beta_{11}$ | -0.40116 | *must be negative* | | | | | | |
| 29 | Solution quadrant | $\beta_{12}$ | 1.45116 | *must be positive* | | | | | | |
| 30 | Solution quadrant | $\beta_{21}$ | 1.42289 | *must be positive* | | | | | | |
| 31 | Solution quadrant | $\beta_{22}$ | -0.36360 | *must be negative* | | | | | | |
| 32 | | EQUATIONS | | | | | | | | |
| 33 | Value matching 1 | EQ 29 | 0.000 | (C24*C19^C28*C26^C29+C19/C5-C26/C6-C25*C19^C30*C26^C31+C15) | | | | | | |
| 34 | Value matching 2 | EQ 30 | 0.000 | (C25*C20^C30*C27^C31-C24*C20^C28*C27^C29-C20/C5+C27/C6+C16) | | | | | | |
| 35 | Smooth pasting 1A | EQ 31 | 0.000 | (C28*C24*(C19^(C28-1))*C26^C29+1/C5-C30*C25*(C19^(C30-1))*C26^C31) | | | | | | |
| 36 | Smooth pasting 1B | EQ 32 | 0.000 | (C29*C24*C19^C28*(C26^(C29-1))-1/C6-C31*C25*C19^C30*(C26^(C31-1))) | | | | | | |
| 37 | Smooth pasting 2A | EQ 33 | 0.000 | (C30*C25*(C20^(C30-1))*C27^C31-C28*C24*(C20^(C28-1))*C27^C29-1/C5) | | | | | | |
| 38 | Smooth pasting 2B | EQ 34 | 0.000 | (C31*C25*C20^C30*(C27^(C31-1))-C29*C24*C20^C28*(C27^(C29-1))+1/C6) | | | | | | |
| 39 | Solution quadrant 1 | EQ 26 | 0.000 | | | | | | | |
| 40 | Solution quadrant 2 | EQ 28 | 0.000 | | | | | | | |
| 41 | | Sum | 0.000 | **Solver** | | | | | | |
| 42 | | SOLVER: SET C41=0, CHANGING C24:C31 | | | | | | | | |
| 43 | | SPREAD | 97.04 | | | | | | | |
| 44 | EQ 26 | 0.5*C7^2*C28*(C28-1)+0.5*C8^2*C29*(C29-1)+C9*C7*C8*C28*C29+C28*(C10-C5)+C29*(C10-C6)-C10 | | | | | | | | |
| 45 | EQ 28 | 0.5*C7^2*C30*(C30-1)+0.5*C8^2*C31*(C31-1)+C9*C7*C8*C30*C31+C30*(C10-C5)+C31*(C10-C6)-C10 | | | | | | | | |
| 46 | $V_1$(x,y) | (C24*C3^C28*C4^C29+C3/C5-C4/C6) | | | | | | | | |
| 47 | $V_2$(x,y) | (C25*C3^C30*C4^C31) | | | | | | | | |

## Numerical Illustrations

Figure 4 is an Excel spreadsheet of the simultaneous solution of the ten equations, assuming $x_{12}=x_{21}$, and deriving the trigger for cost $y_{12}>x_{12}$ that would justify suspension, and the trigger for cost $y_{21}<x_{21}$ that would justify re-starting operations. Note the spreads are very similar to the spreads for the best of two outputs, when the operating cost is constant, due to the similar power function parameter values, but the asset value in operating mode is much lower.

Figure 5 shows the sensitivity of the spreads to changes in the output price volatility, which is similar to Figure 2.

Figure 5

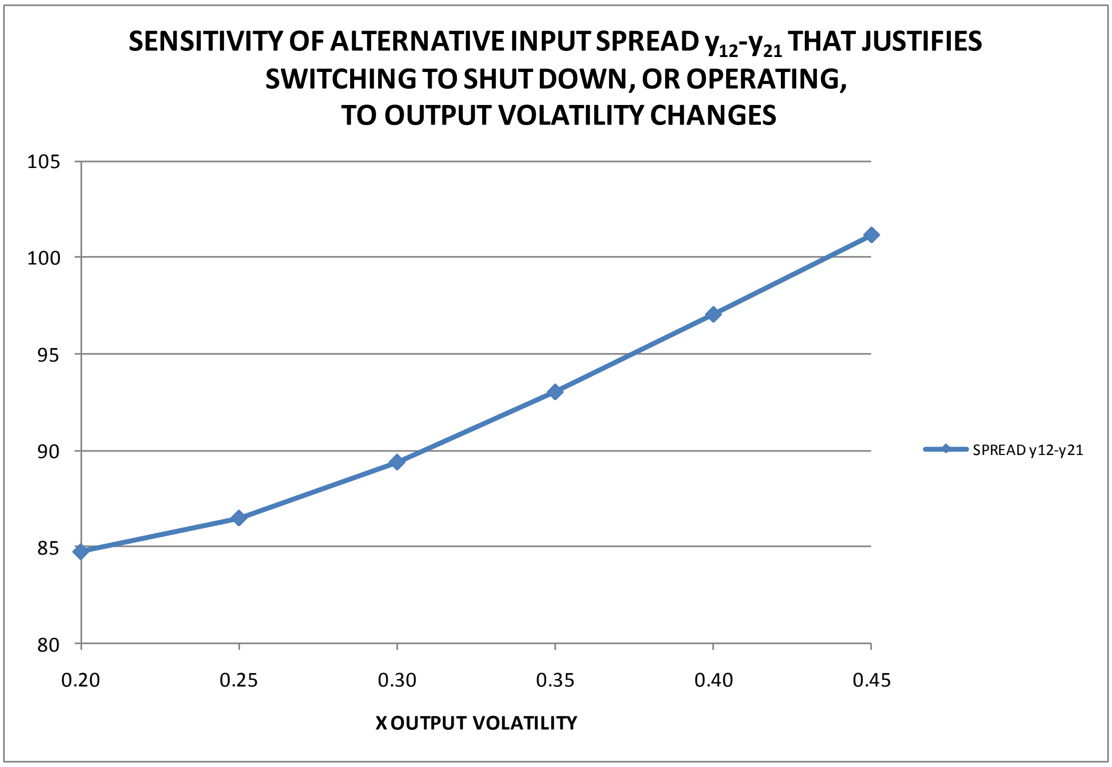

Figure 6

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **SINGLE American Perpetual IN-OUT SWITCH Option Non-homogeneous** | | | | | | | |
| 2 | | INPUT | | | IN_OUT_JAN2011 | | | |
| 3 | PRICE | x | 100 | | | | | |
| 4 | COST | y | 50 | | | | | |
| 5 | Convenience yield of x | $\delta_x$ | 0.04 | | | | | |
| 6 | Convenience yield of y | $\delta_y$ | 0.04 | | | | | |
| 7 | Volatility of x | $\sigma_x$ | 0.40 | | | | | |
| 8 | Volatility of y | $\sigma_y$ | 0.30 | | | | | |
| 9 | Correlation x with y | $\rho$ | 0.50 | | | | | |
| 10 | Risk-free interest rate | r | 0.05 | | | | | |
| 11 | Switching cost from x to y | $S_{12}$ | 50 | | | | | |
| 12 | Switching boundary OP to SHUT | $x_{12}$ | 100 | | | | | |
| 13 | | SOLUTION | | OPTION | OPERATING | | | |
| 14 | Asset value in operating mode | $\mathbf{V_1(x,y)}$ | **1630.84** | **380.84** | **1250.00** | | | |
| 15 | | A | 9.89 | | | | | |
| 16 | Switching boundary OP to SHUT | $y_{12}$ (x) | **337.04** | | | | | |
| 17 | Solution quadrant | $\beta_{11}$ | -0.42545 | *must be negative* | | | | |
| 18 | Solution quadrant | $\beta_{12}$ | 1.43396 | *must be positive* | | | | |
| 19 | | EQUATIONS | | | | | | |
| 20 | Value matching 1 | EQ 35 | 0.000 | | | | | |
| 21 | Smooth pasting 1A | EQ 36 | 0.000 | | | | | |
| 22 | Smooth pasting 1B | EQ 37 | 0.000 | | | | | |
| 23 | Solution quadrant 1 | EQ 38 | 0.000 | | | | | |
| 24 | | Sum | 0.000 | **Solver** | | | | |
| 25 | | SOLVER: SET C24=0, CHANGING C15:C18 | | | | | | |
| 26 | SPREAD | | 287.04 | | | | | |
| 27 | $V_1$(x,y) | (C15*C3^C17*C4^C18+C3/C5-C4/C6) | | | | | | |
| 28 | EQ 35 | (C15*C12^C17*C16^C18+C12/C5-C16/C6+C11) | | | | | | |
| 29 | EQ 36 | (C17*C15*(C12^(C17-1))*C16^C18+1/C5) | | | | | | |
| 30 | EQ 37 | (C18*C15*C12^C17*(C16^(C18-1))-1/C6) | | | | | | |
| 31 | EQ 38 | 0.5*C7^2*C17*(C17-1)+0.5*C8^2*C18*(C18-1)+C9*C7*C8*C17*C18+C17*(C10-C5)+C18*(C10-C6)-C10 | | | | | | |

Figure 6 shows that the single switching boundary is 191% greater than for multiple switching, with similar parameter values. The asset value $V_{1S}$ is 5% lower compared to multiple switching, because the option to shut without any re-starting is almost 20% lower than the multiple switching option.

## 4. MULTIPLE INPUT SWITCHING WITH CONSTANT SWITCHING COSTS

### *4.1 Assumptions*

Consider a flexible facility which can use one of two different inputs by switching between operating modes. Assume the prices of the two inputs x and y, are stochastic and correlated and follow gBm:

$$dx = (\mu_x - \delta_x)x\,dt + \sigma_x\,x\,dz_x \tag{39}$$

$$dy = (\mu_y - \delta_y)y\,dt + \sigma_y\,y\,dz_y \tag{40}$$

with the notations:

μ Required return on the input

δ Convenience yield of the input

σ Volatility of the input

dz Wiener process (stochastic element)

ρ Correlation between the two input prices: $dz_x\, dz_y$ / dt

The instantaneous cash flow in each operating mode is the unit output price less the respective price of the input, assuming production of one (equivalent) unit per annum, (p-x) in operating mode '1' and (p-y) in operating mode '2'. A switching cost of $S_{12}$ is incurred when switching from operating mode '1' to '2', and $S_{21}$ for switching back. The appropriate discount rate is r for non- stochastic elements, such as constant output prices. For convenience and simplicity, assume that the appropriate discount rate for stochastic variables is δ which is equal to μ-r.

Further assumptions are that the output price is deterministic and constant, the lifetime of the asset is infinite, and the company is not restricted in the input mix choice because of quality requirements or operating efficiency. Moreover, the typical assumptions of real options theory apply, with interest rates, convenience yields, volatilities and correlation constant over time.

### *4.2 Quasi-analytical Solution for Continuous Switching*

The asset value with opportunities to continuously switch between the two operating modes is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let $V_1$ be the asset value in operating mode '1', using input x, and $V_2$ the asset value in operating mode '2', using input y accordingly. The switching options depend on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by the following PDE, allowing for different output prices using the different inputs:

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_X\sigma_Y xy\frac{\partial^2 V_1}{\partial x \partial y} + \left(r-\delta_X\right)x\frac{\partial V_1}{\partial x} + \left(r-\delta_Y\right)y\frac{\partial V_1}{\partial y} - rV_1 + \left(p_x - x\right) = 0 \tag{41}$$

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_X\sigma_Y xy\frac{\partial^2 V_2}{\partial x \partial y} + \left(r-\delta_X\right)x\frac{\partial V_2}{\partial x} + \left(r-\delta_Y\right)y\frac{\partial V_2}{\partial y} - rV_2 + \left(p_y - y\right) = 0 \tag{152}$$

Adkins and Paxson (2011) derive a quasi-analytical solution for a similar type of two-factor non-homogeneous problem. For two inputs, the PDE are satisfied by the following general solutions:

$$V_1\left(x,y\right)=\frac{p_x}{r}-\frac{x}{\delta_x}+A\,x^{\beta_{11}}\,y^{\beta_{12}} \tag{43}$$

where $\beta_{11}$ and $\beta_{12}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1)+\tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1)+\rho\sigma_x\sigma_y\beta_{11}\beta_{12}+\beta_{11}(r-\delta_x)+\beta_{12}(r-\delta_y)-r=0 \tag{44}$$

and

$$V_2\left(x,y\right)=\frac{p_y}{r}-\frac{y}{\delta_y}+B\,x^{\beta_{21}}\,y^{\beta_{22}} \tag{45}$$

where $\beta_{21}$ and $\beta_{22}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{21}(\beta_{21}-1)+\tfrac{1}{2}\sigma_y^2\beta_{22}(\beta_{22}-1)+\rho\sigma_x\sigma_y\beta_{21}\beta_{22}+\beta_{21}(r-\delta_x)+\beta_{22}(r-\delta_y)-r=0 \tag{46}$$

Since the option to switch from x to y increases with x and decreases with y, $\beta_{11}$ must be positive and $\beta_{12}$ negative. Likewise, $\beta_{21}$ must be negative and $\beta_{22}$ positive. Switching between the operating modes always depends on the level of both x and y. At the switching points ($x_{12}$, $y_{12}$) and ($x_{21}$, $y_{21}$), the asset value in the current operating mode must be equal to the asset value in the alternative operating mode net of switching cost. Allowing for possible different prices for output produced by x and y, these value matching conditions are:

$$A\,x_{12}^{\ \beta_{11}}\,y_{12}^{\ \beta_{12}}+\frac{p_x}{r}-\frac{x_{12}}{\delta_x}--=B\,x_{12}^{\ \beta_{21}}\,y_{12}^{\ \beta_{22}}+\frac{p_y}{r}-\frac{y_{12}}{\delta_y}-S_{12} \tag{47}$$

$$A\,x_{21}^{\ \beta_{11}}\,y_{21}^{\ \beta_{12}}+\frac{p_x}{r}-\frac{x_{21}}{\delta_x}-S_{21}=B\,x_{21}^{\ \beta_{21}}\,y_{21}^{\ \beta_{22}}+\frac{p_y}{r}-\frac{y_{21}}{\delta_y} \tag{16}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11}\,A\,x_{12}^{\ \beta_{11}-1}\,y_{12}^{\ \beta_{12}}-\frac{1}{\delta_x}=\beta_{21}\,B\,x_{12}^{\ \beta_{21}-1}\,y_{12}^{\ \beta_{22}} \tag{49}$$

$$\beta_{12}\,A\,x_{12}^{\ \beta_{11}}\,y_{12}^{\ \beta_{12}-1}=\beta_{22}\,B\,x_{12}^{\ \beta_{21}}\,y_{12}^{\ \beta_{22}-1}-\frac{1}{\delta_y} \tag{50}$$

$$\beta_{11} A x_{21}^{\beta_{11}-1} y_{21}^{\beta_{12}} - \frac{1}{\delta_x} = \beta_{21} B x_{21}^{\beta_{21}-1} y_{21}^{\beta_{22}} \tag{51}$$

$$\beta_{12} A x_{21}^{\beta_{11}} y_{21}^{\beta_{12}-1} = \beta_{22} B x_{21}^{\beta_{21}} y_{21}^{\beta_{22}-1} - \frac{1}{\delta_y} \tag{52}$$

There are only 8 equations for 10 unknowns, $\beta_{11}$, $\beta_{12}$, $\beta_{21}$, $\beta_{22}$, A, B, $x_{12}$, $y_{12}$, $x_{21}$, $y_{21}$, so there is no completely analytical solution. Yet, for every value of x, there has to be a corresponding value of y when switching should occur, ($x_{12}$, $y_{12}$) from x to y and ($x_{21}$, $y_{21}$) from y to x. So a quasi-analytical solution can be found by assuming values for x, which then solves the set of simultaneous equations for all remaining variables, given that $x = x_{12} = x_{21}$. This procedure is repeated for many values of x, providing the corresponding option values and the switching boundaries.

### *4.3 Quasi-analytical Solution for One-Way Switching*

The solution for the asset value with a one-way switching option from the above model with continuous switching is straight-forward. Assuming one starts with input x[2], the American perpetual option to switch from x to y can be determined. Applying the same solution procedure as before, a quasi-analytical solution is obtained.

$$A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}} + \frac{p_x}{r} - \frac{x_{12}}{\delta_x} = \frac{p_y}{r} - \frac{y_{12}}{\delta_y} - S_{12} \tag{52}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}^{\beta_{11}-1} y_{12}^{\beta_{12}} - \frac{1}{\delta_x} = 0 \tag{53}$$

$$\beta_{12} A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}-1} + \frac{1}{\delta_y} = 0 \tag{54}$$

where $\beta_{11}$ and $\beta_{12}$ satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1) + \tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1) + \rho\sigma_x\sigma_y\beta_{11}\beta_{12} + \beta_{11}(r-\delta_x) + \beta_{12}(r-\delta_y) - r = 0 \tag{55}$$

The characteristic root equation (55) together with value matching condition (52) and smooth pasting conditions (53) and (54) represents the system of 4 equations, while there are 5 unknowns, $\beta_{11}$, $\beta_{12}$, A, $x_{12}$, $y_{12}$.

[2] Adkins and Paxson (2011) allowed starting either with x or y, depending on whether $V_1$ is greater than $V_2$ (in which starting with x is logical, if feasible).

## Numerical Illustrations

Here are illustrative results for the multiple and single output switch models, assuming current operating costs are half of current gross revenue for each output. Figure 7 shows that the option factors A and B are positive, $\beta_{11}$ and $\beta_{22}$ are negative and $\beta_{12}$ and $\beta_{21}$ are positive, thereby fulfilling the requirements from the theoretical model. The system of value matching conditions, smooth pasting conditions and characteristic root equations is fully satisfied.

Figure 7

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Multiple American Perpetual Input Switching Option** | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | INPUT x | x | 50 | | | | | | |
| 4 | INPUT y | y | 50 | | | | | | |
| 5 | Convenience yield of x | $\delta_x$ | 0.04 | | | | | | |
| 6 | Convenience yield of y | $\delta_y$ | 0.04 | | | | | | |
| 7 | Volatility of x | $\sigma_x$ | 0.40 | | | | | | |
| 8 | Volatility of y | $\sigma_y$ | 0.30 | | | | | | |
| 9 | Correlation x with y | $\rho$ | 0.50 | | | | | | |
| 10 | Risk-free interest rate | r | 0.05 | | | | | | |
| 11 | Output price for x | $p_x$ | 100 | | | | | | |
| 12 | Output price for y | $p_y$ | 100 | | | | | | |
| 13 | | | | | | | | | |
| 14 | | | | | | | | | |
| 15 | Switching cost from x to y | $S_{12}$ | 50 | | | | | | |
| 16 | Switching cost from y to x | $S_{21}$ | 70 | | | | | | |
| 17 | PV of revenues using x | X | 750 | | | | | | |
| 18 | PV of revenues using y | Y | 750 | | | | | | |
| 19 | Switching boundary X to Y | $x_{12}$ | 50 | | | | | | |
| 20 | Switching boundary Y to X | $x_{21}$ | 50 | | | | | | |
| 21 | | SOLUTION | | OPTION VALUE | | | | | |
| 22 | Asset value in operating mode '1' | **$V_1$(x,y)** | **1,380.66** | **631** | | | | | |
| 23 | Asset value in operating mode '2' | **$V_2$(x,y)** | **1,362.14** | **612** | | | | | |
| 24 | | A | 9.25 | | | | | | |
| 25 | | B | 8.79 | | | | | | |
| 26 | Switching boundary x to y | $y_{12}$ (x) | **22.29** | | | | | | |
| 27 | Switching boundary y to x | $y_{21}$ (x) | **84.48** | | | | | | |
| 28 | Solution quadrant | $\beta_{11}$ | 1.420 | *must be positive* | | | | | |
| 29 | Solution quadrant | $\beta_{12}$ | -0.340 | *must be negative* | | | | | |
| 30 | Solution quadrant | $\beta_{21}$ | -0.380 | *must be negative* | | | | | |
| 31 | Solution quadrant | $\beta_{22}$ | 1.464 | *must be positive* | | | | | |
| 32 | | EQUATIONS | | | | | | | |
| 33 | Value matching 1 | EQ47 | 0.000 | | | | | | |
| 34 | Value matching 2 | EQ48 | 0.000 | | | | | | |
| 35 | Smooth pasting 1A | EQ49 | 0.000 | | | | | | |
| 36 | Smooth pasting 1B | EQ50 | 0.000 | | | | | | |
| 37 | Smooth pasting 2A | EQ51 | 0.000 | | | | | | |
| 38 | Smooth pasting 2B | EQ52 | 0.000 | | | | | | |
| 39 | Solution quadrant 1 | EQ44 | 0.000 | | | | | | |
| 40 | Solution quadrant 2 | EQ46 | 0.000 | | | | | | |
| 41 | | Sum | 0.000 | | | | | | |
| 42 | | SOLVER: SET C40=0, CHANGING C24:C31 | | | | | | | |
| 43 | EQ47 | (C24*C19^C28*C26^C29-C19/C5+C11/C10-C25*C19^C30*C26^C31+C26/C6-C12/C10+C15) | | | | | | | |
| 44 | EQ48 | (C24*C20^C28*C27^C29-C20/C5+C11/C10-C25*C20^C30*C27^C31+C27/C6-C12/C10-C16) | | | | | | | |
| 45 | EQ49 | (C28*C24*C19^(C28-1)*C26^C29-1/C5-C30*C25*C19^(C30-1)*C26^C31) | | | | | | | |
| 46 | EQ50 | (C29*C24*C19^C28*C26^(C29-1)-C31*C25*C19^C30*C26^(C31-1)+1/C6) | | | | | | | |
| 47 | EQ51 | (C28*C24*C20^(C28-1)*C27^C29-1/C5-C30*C25*C20^(C30-1)*C27^C31) | | | | | | | |
| 48 | EQ52 | (C29*C24*C20^C28*C27^(C29-1)-C31*C25*C20^C30*C27^(C31-1)+1/C6) | | | | | | | |
| 49 | EQ44 | 0.5*C7^2*C28*(C28-1)+0.5*C8^2*C29*(C29-1)+C9*C7*C8*C28*C29+C28*(C10-C5)+C29*(C10-C6)-C10 | | | | | | | |
| 50 | EQ46 | 0.5*C7^2*C30*(C30-1)+0.5*C8^2*C31*(C31-1)+C9*C7*C8*C30*C31+C30*(C10-C5)+C31*(C10-C6)-C10 | | | | | | | |
| 51 | SPREAD | | **62.20** | | | | | | |
| 52 | $V_1$(x,y) | -C3/C5+C11/C10+C24*C3^C28*C4^C29 | | | | | | | |
| 53 | $V_2$(x,y) | -C4/C6+C12/C10+C25*C3^C30*C4^C31 | | | | | | | |

The asset values are given in both operating modes, $V_1$ and $V_2$, and the level of y is indicated when it is optimal to switch from x to y ($y_{12}$) and vice versa ($y_{21}$). With x and y having the same initial values and the same convenience yields, the asset value with no switching is identical in both operating modes when the output price is the same. Higher output prices increase the asset value. The spread between $y_{12}$ and $y_{21}$ is due to switching costs and stochastic elements, and increases with high volatilities and low correlation, following real options theory. It should be noted that changing x also changes the switching boundaries $y_{12}$ and $y_{21}$, and that the switching boundaries $x_{12}$ and $x_{21}$ for a given level of y can be determined in a similar way. The fact that $y_{12}$ and $y_{21}$ are not exactly symmetrical to x = 50 is primarily due to the log normality of the prices, and further due to $S_{12} \neq S_{21}$ and $\sigma_x \neq \sigma_y$.

Figure 8

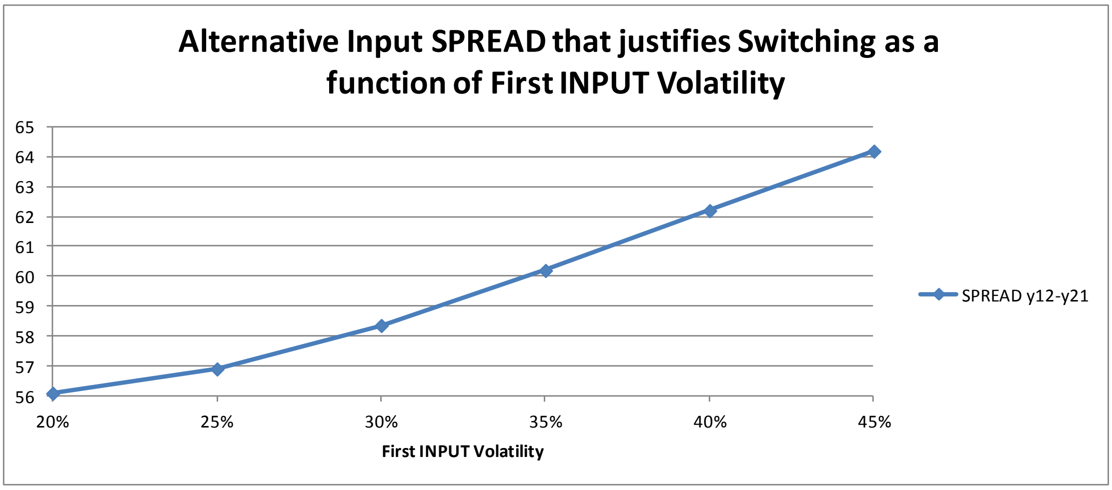


Figure 8 illustrates the sensitivity of the switching boundaries of the quasi-analytical solution for continuous switching to changes in x input volatility. Switching boundaries are further apart when volatilities are higher. This is consistent with general real option theory because uncertainty is taken into account which delays switching in order to gain more information.

When switching is only possible from x to y but not vice versa, the switching trigger $y_{12S}$ is much lower as shown in Figure 9 because the decision cannot be reversed. The asset value $V_{1S}$ is lower compared to multiple switching.

Figure 9

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Continuous American Perpetual SINGLE SWITCH Option** | | | | | | | |
| 2 | ONE WAY SWITCH FROM INPUT x TO y | | | | | | | |
| 3 | INPUT x | x | 50 | | | | | |
| 4 | INPUT y | y | 50 | | | | | |
| 5 | Convenience yield of x | $\delta_x$ | 0.04 | | | | | |
| 6 | Convenience yield of y | $\delta_y$ | 0.04 | | | | | |
| 7 | Volatility of x | $\sigma_x$ | 0.40 | | | | | |
| 8 | Volatility of y | $\sigma_y$ | 0.30 | | | | | |
| 9 | Correlation x with y | $\rho$ | 0.50 | | | | | |
| 10 | Risk-free interest rate | r | 0.05 | | | | | |
| 11 | Output price for x | $p_x$ | 100 | | | | | |
| 12 | Output price for y | $p_y$ | 100 | | | | | |
| 13 | Switching cost from x to y | $S_{12}$ | 50 | | | | | |
| 14 | | | | | | | | |
| 15 | PV of revenues using x | X | 750 | | | | | |
| 16 | PV of revenues using y | Y | 750 | | | | | |
| 17 | Switching boundary x to y | $x_{12}$ | 50 | | | | | |
| 18 | | | | | | | | |
| 19 | | SOLUTION | | OPTION VALUE | | | | |
| 20 | Asset value in operating mode '1' | $\mathbf{V_1(x,y)}$ | **1,284.22** | **534.22** | | | | |
| 21 | Asset value in operating mode '2' | $\mathbf{V_2(x,y)}$ | **750.00** | | | | | |
| 22 | | A | 8.55 | | | | | |
| 23 | Switching boundary x to y | $y_{12}$ (x) | **12.87** | | | | | |
| 24 | Solution quadrant | $\beta_{11}$ | 1.4232 | *must be positive* | | | | |
| 25 | Solution quadrant | $\beta_{12}$ | -0.3663 | *must be negative* | | | | |
| 26 | | EQUATIONS | | | | | | |
| 27 | Value matching 1 | EQ 52 | 0.000 | | | | | |
| 28 | Smooth pasting 1A | EQ 53 | 0.000 | | | | | |
| 29 | Smooth pasting 1B | EQ 54 | 0.000 | | | | | |
| 30 | Solution quadrant 1 | EQ 55 | 0.000 | | | | | |
| 31 | | Sum | 0.000 | | | | | |
| 32 | | SOLVER: SET C31=0, CHANGING C22:C25 | | | | | | |
| 33 | EQ 52 | C22*C17^C24*C23^C25-C17/C5+C11/C10+C23/C6-C12/C10+C13 | | | | | | |
| 34 | EQ 53 | C24*C22*C17^(C24-1)*C23^C25-1/C5 | | | | | | |
| 35 | EQ 54 | C25*C22*C17^C24*C23^(C25-1)+1/C6 | | | | | | |
| 36 | EQ 55 | 0.5*C7^2*C24*(C24-1)+0.5*C8^2*C25*(C25-1)+C9*C7*C8*C24*C25+C24*(C10-C5)+C25*(C10-C6)-C10 | | | | | | |
| 37 | SPREAD | | **37.13** | | | | | |
| 38 | $V_1$(x,y) | -C3/C5+C11/C10+C22*C3^C24*C4^C25 | | | | | | |
| 39 | $V_2$(x,y) | -C4/C6+C12/C10 | | | | | | |

## 5 Policy and Strategy Implications

There are a number of stakeholders shown in Figure 10 whose best decisions should be based on these switching models.

Figure 10

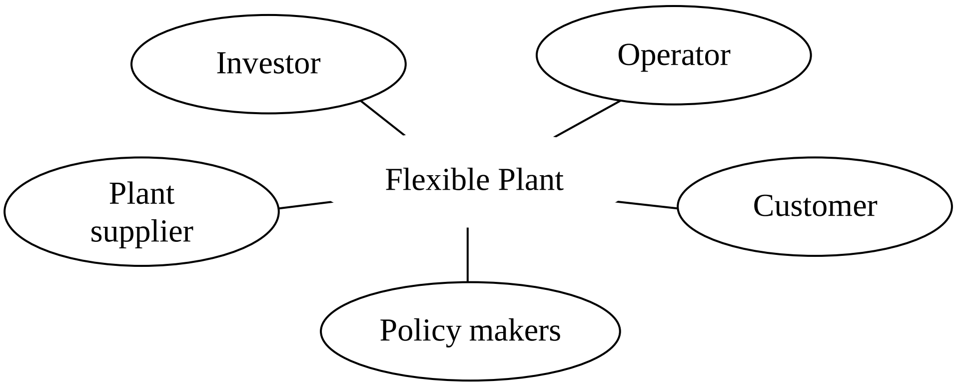


Figure 11

| | TWO OUTPUTS | IN-OUT | TWO INPUTS |
|---|---|---|---|
| MULTIPLE | | | |
| PRODUCTION | 1500.00 | 1250.00 | 750.00 |
| V1 | 2780.37 | 1718.27 | 1380.66 |
| V2 | 2781.76 | 1649.16 | 1362.14 |
| SPREAD | 97.04 | 97.04 | 62.20 |
| | | | |
| SINGLE | | | |
| PRODUCTION | 1500.00 | 1250.00 | 750.00 |
| V1 | 2528.98 | 1630.84 | 1284.22 |
| V2 | 1500.00 | 0.00 | 750.00 |
| SPREAD | 237.04 | 287.04 | 37.13 |
| | | | |
| Multiple/Single | 109.94% | 105.36% | 107.51% |

**Investors**

As shown in Figure 11, the real option value of all of these flexible facilities is substantially greater than the present value of current production (= inflexible facilities), at the current assumed input and output price levels. Note the focus of alert investors is on choosing the appropriate model and on forecasting input and output price volatilities and correlations. A myopic investment analyst using net present values will probably undervalue flexible plants. Analysts may not have access to plant operating or switching costs, or indeed knowledge of any flexibility inherent in existing facilities, due conceivably to inadequate accounting disclosures, not currently required by accounting standard setting committees. Of course, realistic analysts may doubt that the chief option managers of flexible facilities will be aware of the potential optionality, or indeed make switches at appropriate times, so the Marshallian values might reflect a realistic allowance for management shortfalls.

### Chief Real Options Manager

The alert chief options manager ("CROM") is aware of input and output switching opportunities, the amount of switching costs, and periodically observes input and output prices, convenience yields (or proxies), updates expected volatilities and correlations, and so updates Figure 11 appropriately. Observed current spreads between input/output prices are compared to the updated triggers for switching, perhaps based on simple approximate linear rules over short or stable periods. Naturally part of the appropriate compensation for the CROM should be based on awareness of these opportunities, and performance in making actual input and output switches at appropriate times.

Originally, the CROM would have calculated the value of a flexible plant $V_1$ or $V_2$, compared to an inflexible facility, which also indicates the warranted extra investment cost for facility flexibility. It would not be difficult to consider trade-offs for any deterministic lower efficiency due to the flexibility capacity.

### Plant Suppliers

Originally, suppliers of facilities to the CROM would have calculated the value of a flexible plant $V_1$ or $V_2$, compared to an inflexible facility, which also indicates the warranted extra investment price that could be charged for facility flexibility. With the illustrated parameter values, a hypothetical multiple switch facility is worth only some 5-10% more than a single switch plant, but much more than an inflexible facility. In designing flexible facilities, it would not be difficult to consider trade-offs for any lower efficiency due to the flexibility capacity against increased building costs.

### Customers

Output customers may be aware of the limitations, or capacities, of producers to switch to higher price products, opportunistically, or to alternative lower price inputs when appropriate. Input suppliers may become cautious with buyers, who switch sources optimally. Other customers might seek long-term agreements mitigating the shifts in output and input prices implied in using real option approaches for operating flexible facilities.

### Policy Makers

Taxpayers beware. There will be national producers without flexible facilities, or not aware of needing to change output prices, and input sources, as the economic environment changes. Those producers priced out of the market will seek government barriers for other producers, or input/output subsidies as conditions change.

## 6 Applications

Flexibility between outputs and inputs is particularly relevant in volatile commodity markets, or where free trade allows new entrants, cheaper inputs, or more valuable outputs. Think of the many applications for substitute outputs, substitute inputs, or alternative inputs and outputs. Dockendorf and Paxson (2011) examine further processed chemical products as essentially output alternatives. They note alternative uses of other types of facilities, such as multiuse sports or entertainment or educational facilities, transportation vehicles for passengers or cargo, rotating agricultural crops, and solar energy used for electricity or water desalination. Adkins and Paxson (2011) note there are numerous energy switching opportunities, such as palm or rape oil in biodiesel production, gas-oil-hydro-coal in electricity generation, that are reciprocal energy input switching options. There are several examples of stochastic output and input prices, such as the "crack" spread for gasoline-heating oil as outputs for crude oil refineries, the "crush" spread for soya meal and soya oil as outputs for soya bean refineries, and ethanol the output for corn processing facilities.

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EXERCISE 12.1

George Gamble owns and operates the HBS renewable biodiesel facility, which can be switched once (at a cost of \$ .50) between canola and palm oil inputs. He is currently using canola at \$1 per unit, as is palm oil, both have volatility of 20%, .5 correlated, convenience yield of 4%, r=5%, and output sells for \$3 per unit. If palm oil is still $\$1=X_{22}$, at what increased canola price should he switch to palm oil ($X_{12}$)? George figures that $\beta_{11}=2.01$, $\beta_{12}=-.93$, and A=6.07. HINT: $y_{12}/x_{12}=-\beta_{12}/\beta_{11}$. What is value of this facility?

EXERCISE 12.2

Eventually Marianne switches to the aged, kind Colonel Brandon (annual income £ 100=y) over the exciting, handsome Willoughby (ignoring his real marriage options and gambling debts, income £$70=x_{12}$) even though the switching costs ( £100 in emotional pain) are great. Both are volatile (20%), are 100% negatively correlated, are not very convenient (4%), and their fortunes are invested in gilts which yield 5%. The Colonel calculates that $\beta_{11}=-.357$, $\beta_{12}=1.378$, $y_{12}/x_{12}=-\beta_{12}/\beta_{11}$, and $A=(-1/\delta_x)/(\beta_{11}*x_{12}^{(\beta 11-1)} * y_{12}^{\beta 12})$. Is she right even if she receives her husband's income? HINT: see EQ 5, page 5.

EXERCISE 12.3

Julia Smith, who is nearly immortal, is taking the Real Options course (RO), and believes it offers the same career benefit (1 per annum) as Investment Analysis (IA) (1). Both are very easy with no cost or effort, but both are uncertain ($\sigma_{RO}=\sigma_{IA}=10\%$), $\delta_{RO}=\delta_{IA}=5\%$, $\rho=0$, r=5%. Professors Great and Smart have agreed that Julia may switch courses once anytime for a small fee of .2. They report that $\beta_{11}=-1.78$ and $\beta_{12}=2.80$. What IA benefits would justify switching ? The switching boundary equals $-\beta_{12}/\beta_{11}$., and $A=(-1/\delta_x)/(\beta_{11}*y_{12}\wedge\beta_{12})$. What is the current value of taking RO? HINT: see EQ 5, page 5.

PROBLEM 12.4

Lucy Steele is secretly engaged to Edward Ferrar, who has a yearly income of ₤10,000. When Edward's mother learns of this engagement, she transfers his inheritance to the younger brother Robert, who is not handsome, so the cost to Lucy of switching to Robert is significant ( ₤ XXX). Supposing that Edward will eventually obtain ₤7000 annually from Colonel Brandon, would a switching cost of ₤ 3000 justify Lucy's switch, if this can be forever delayed? Both Edward and Robert are volatile (20%), not correlated, are not very convenient (4%), and their fortunes are invested in gilts which yield 5%.

PROBLEM 12.5

George Gamble owns and operates the HBS renewable biodiesel facility, which can switch back and forth between canola and palm oil inputs. He is currently using canola at $1 per unit, as is palm oil, both have volatility of 20%, .5 correlated, convenience yield of 4%, r=5%, switching cost $ .50 and output sells for $3 per unit. When should he switch to palm oil, and then back to canola? What is HBS worth?

PROBLEM 12.6

Julia Smith, who is nearly immortal, believes the Real Option course (RO) offers a career benefit of 10 per annum compared to Investment Analysis (IA) (8). Both are hard courses with a cost of 1. Both are uncertain ($\sigma_{RO}=\sigma_{IA}=10\%$), $\delta_{RO}=\delta_{IA}=5\%$, $\rho=0$, r=5%. Professors Great and Smart have agreed that Julia may switch courses back and forth anytime for a fee of 1. Julia believes the perceived value of these courses to her will fluctuate throughout her job interviews. What IA benefits would justify switching, or once switched what RO benefits would justify switching back. What is the current value of taking RO?